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Dressing method based on the homogeneous Fredholm equation: quasilinear PDEs in multidimensions

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Abstract

In this paper we develop a dressing method for constructing and solving some classes of matrix quasilinear partial differential equations (PDEs) in arbitrary dimensions. This method is based on a homogeneous integral equation with a nontrivial kernel, which allows one to reduce the nonlinear PDEs to systems of non-differential (algebraic or transcendental) equations for the unknown fields. In the simplest examples, the above dressing scheme captures matrix equations integrated by the characteristics method and nonlinear PDEs associated with matrix Hopf–Cole transformations.

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1. Introduction

Since the pioneering work [1] on the Korteweg–de Vries equation [2], completely integrable nonlinear partial differential equations (PDEs) have been intensively studied during the last decades, and large classes of integrable PDEs have been found, such as the so-called S-integrable systems (or soliton equations, whose integration scheme involves the solution of a linear integral equation) [3, 4], and the so-called C-integrable equations (integrated by simpler transformations, such as the Hopf–Cole transformation for the Burgers equation) [5]. Much effort has been devoted to the study of direct techniques to construct and solve nonlinear PDEs. One of the most powerful of such techniques is the dressing method, originally developed for (1+1)- and (2+1)-dimensional S-integrable models [6–9] (see also [10]). Multidimensional generalizations of it have also been developed [11–14], allowing us to integrate special classes of higher dimensional nonlinear PDEs. Nonlinear dressing methods for PDEs in arbitrary dimensions associated with vector fields are also known [15, 16]. All the above dressing formalisms are based on the hypothesis that, for given spectral data, the spectral function can be uniquely constructed from the relevant integral equation, i.e., the kernel of the corresponding integral operator is empty.

Recently, a new version of the dressing method has appeared, based on integral operators with a nontrivial kernel [17]. This assumption removes most of the restrictions on the dimensionality of the space of analytic solutions of the constructed PDEs, and the solutions of the n-dimensional PDEs constructed in [17] are parametrized by arbitrary spectral functions of $(n-2)$ variables (full integrability would be achieved if the variables were $(n-1)$). The structure of the nonlinear PDEs has been simplified and the space of analytic solutions has been enriched in [18], where multidimensional dressing operators [19], which are not effective in the classical case, have been used. The solutions of the n-dimensional nonlinear PDEs constructed there are parametrized by arbitrary spectral functions of $(n-1)$ variables, but full integrability has not been achieved even there, since these spectral functions are not in the right number.

In this paper we develop another variant of the dressing method, based on a homogeneous integral equation with a nontrivial kernel, allowing one to reduce certain classes of nonlinear PDEs in arbitrary dimensions to systems of non-differential (algebraic and/or transcendental) equations, similarly to the method of characteristics [20]. The nonlinear PDEs in arbitrary dimensions isolated by this method are built in terms of 'dressed' first-order operators of the type (4) below. To increase the dimensionality of these PDEs and of their space of analytic solutions, we use multidimensional differential operators first introduced in [18].

Below is the list of the nonlinear PDEs which will be derived and solved in this paper. All equations are $Q \times Q$ matrix equations ($Q \in \mathbb{N}_+$), unless differently specified. Hereafter we write superscripts in parentheses in order to distinguish them from the power notation.

(1) The first-order matrix equation

$$w_{t_1} + \sum_{j=1}^{N} w_{x_j} \rho^{(j)}(w) = [w, T\rho^{(0)}(w)] \tag{1}$$

will be derived in section 3. Here T is any constant matrix, $\rho^{(i)}(w)$ are arbitrary matrix functions representable as positive power series of w, and N is any integer. This matrix equation can also be integrated [21] by the method of characteristics.

(2) A minor modification of the dressing method for equations (1) allows one to construct a class of second-order matrix systems (see section 5) which degenerate, for the simplest choice of the arbitrary functions, to the matrix Burgers equation, linearizable by the Hopf–Cole transformation. A typical example is given by the second-order matrix system,

$$\begin{aligned} &\mathcal{L}_2(w) = [w, S(v^2 - \mathcal{L}_1(v))v], \\ &\mathcal{L}_2(v) + S\mathcal{L}_1^2(v) = S\mathcal{L}_1(v^2) + [S\mathcal{L}_1(v), v] + [v, Sv^2], \end{aligned} \tag{2}$$

subjected to the constraint

$$\mathcal{L}_1(w) = [w, v], \tag{3}$$

where S is any constant diagonal matrix, the differential operators $\mathcal{L}_m$, $m = 1, 2$, are defined as follows:

$$\mathcal{L}_m(f(x)) = f_{t_m}(x) + \sum_{j=1}^{N} f_{x_j}(x)\rho^{(mj)}(w), \tag{4}$$

and $\rho^{(mj)}(w)$ are arbitrary matrix functions representable as positive power series of w.

We remark that many equations of mathematical physics appear as first-order quasilinear PDEs in multidimensions. Therefore it is important to develop efficient methods to isolate and solve, among these equations, integrable and/or partially integrable cases.

The matrix equation (1), a remarkable example of an integrable system of PDEs in arbitrary dimensions, is the natural matrix generalization of the classical examples of physically relevant scalar equations integrable by the method of characteristics. A systematic study of the matrix reductions of (1), with the goal of isolating physically relevant cases, will be the subject of a subsequent paper. Here we only remark that, in the 2×2 matrix reduction

$$w = \begin{pmatrix} v_1 & v_2 \\ v_2 & v_1 \end{pmatrix}, \tag{5}$$

equation (1), with $N = 1$ and $\rho^{(0)} = 0$, coincides [21] with the gas-dynamics equations [20]

$$v_{1t} + v_1 v_{1x_1} + v_2 v_{2x_1} = 0, \qquad v_{2t} + v_2 v_{1x_1} + v_1 v_{2x_1} = 0. \tag{6}$$

We also remark that other vector generalizations of scalar equations integrable by the method of characteristics have been studied during the last 20 years, by different methods, in a set of papers [22–24].

The system of second-order equations (2), constructed in terms of the two first-order multidimensional operators (4), presents mathematical features in common with equations integrable by the method of characteristics, and with equations integrable by the Hopf–Cole transformation; therefore it is conceptually similar to the equations, introduced in [14], presenting mathematical features of both S-integrable and C-integrable PDEs. Equations such as (2), for which integrability properties of different types merge together, are interesting prototype examples in the theory of integrable systems.

The paper is organized as follows. In section 2, after a brief review of the classical dressing method (in section 2.1) and of the novel dressing features contained in [17] (in section 2.2), we describe the main features of the general dressing algorithm of this paper. Equations (1) will be derived in section 3. In section 4 we use the dressing algorithm to reduce equations (1) to a system of non-differential equations characterizing the general solution of (1); in particular, we discuss the Cauchy problem. In section 5 we derive equations (2) and the associated general class, discussing their solution space. In section 6 we make some remarks on the overdetermined system of linear PDEs for the associated spectral function. Conclusions are presented in section 7.

2. Basic novelties of the dressing method

To emphasize all the significant novelties and features of our algorithm, in comparison with both the classical dressing method and the new dressing method developed in [17], first, we give a brief review of one of the versions of the classical dressing method for the (2+1)-dimensional N-wave equation (section 2.1), and, second, we describe the modifications introduced in [17] (section 2.2). After such a preliminary overview, we explain the novelties of the algorithm introduced in this paper (section 2.3).

2.1. Brief review of the classical version of the dressing method, and the (2+1)-dimensional N-wave equation

The starting point of one of the versions of the classical dressing method is the linear integral equation

$$\Phi(\lambda; x) = \int \Psi(\lambda, \mu; x) U(\mu; x)\, \mathrm{d}\Omega(\mu) \equiv \hat{\Psi} U(\lambda; x), \tag{7}$$

in the spectral variables $\lambda = (\lambda_1, \ldots, \lambda_{\dim\lambda})$, $\mu = (\mu_1, \ldots, \mu_{\dim\lambda})$, for the unknown matrix function U. The given matrix functions Φ and Ψ are defined by some extra conditions, which fix their dependence on an additional vector parameter $x = (x_1, \dots, x_{\dim x})$, whose components are the independent variables of the associated nonlinear PDEs. Ω is some largely arbitrary scalar measure in the μ-space. Apart from Ω, all the functions appearing in this paper are $Q \times Q$ matrix functions.

We remark that no a priory assumption is made in (7) on the dependence of Ψ on λ (this general starting point has been used, for instance, in [14, 25]), to keep the structure of Ψ as much general as possible. Indeed, although in most of the cases such a dependence is described by a Cauchy kernel, an indication that equation (7) is the manifestation of Riemann–Hilbert and/or $\bar\partial$ analyticity problems, there are examples (see [25] and [14]) in which more general representations appear, indicating that the above analyticity problems could be too restrictive starting points.

2.1.1. Derivation of an N-wave system. The basic assumption underlying all the known classical dressing procedures available in the literature is that the operator $\hat\Psi$ in (7) is uniquely invertible, i.e.,

$$\dim\ker\hat\Psi = 0. \tag{8}$$

The x-dependence is introduced by the matrix equations

$$\Psi_{x_i}(\lambda, \mu; x) = \Phi(\lambda; x) B_i C(\mu; x), \qquad i = 1, \dots, \dim x, \tag{9}$$

showing that the x-derivatives of the kernel Ψ are degenerate matrix functions, another basic feature of all known classical dressing algorithms, where $B_i, i = 1, \dots, \dim x$, are constant diagonal matrices, at most Q of which may be independent. Due to the above degeneracy, the compatibility of equations (9) leads to separate equations for Φ and C,

$$\Phi_{x_i} B_j - \Phi_{x_j} B_i = 0, \qquad i \neq j, \tag{10}$$

$$B_j C_{x_i} - B_i C_{x_j} = 0, \qquad i \neq j, \tag{11}$$

and one equation is the adjoint of the other. Without loss of generality we assume $B_1 = I$, where I is the identity matrix.

Replacing in equation (10) Φ by $\hat\Psi U$, as indicated in (7), and using (9), one obtains the following equation:

$$\hat\Psi L_{ij} U = 0, \tag{12}$$

where

$$L_{ij} U \equiv U_{x_i} B_j - U_{x_j} B_i + U B_i v B_j - U B_j v B_i, \qquad i, j = 1, \dots, \dim x, \quad i \neq j \tag{13}$$

and

$$v(x) \equiv \int C(\lambda; x) U(\lambda; x)\, \mathrm{d}\Omega(\lambda). \tag{14}$$

Then property (8) implies that

$$L_{ij} U(\lambda; x) = 0, i, j = 1, \dots, \dim x, \qquad i \neq j \tag{15}$$

or, explicitly,

$$\begin{aligned} L_{21} U &= U_{x_2} - U_{x_1} B_2 - U[v, B_2] = 0, \\ L_{31} U &= U_{x_3} - U_{x_1} B_3 - U[v, B_3] = 0, \end{aligned} \tag{16}$$

having chosen $j = 1, i = 2, 3$.

This is nothing but the well-known linear overdetermined system corresponding to the N-wave equation in the three variables x_1, x_2, x_3.

Its compatibility condition yields the N-wave system:

$$\left[v_{x_3}, B_2\right] - \left[v_{x_2}, B_3\right] + B_2 v_{x_1} B_3 - B_3 v_{x_1} B_2 - [[v, B_2], [v, B_3]] = 0. \tag{17}$$

It is important to remark that

(i) equation (17) may be derived in a different way, 'saturating the parameter λ' in equations (16) by the integral operator $\int \mathrm{d}\Omega(\lambda) C(\lambda; x)\cdot$, and obtaining the nonlinear system,

$$\begin{aligned} L_{21} v - [B_2, v_1] &= v_{x_2} - v_{x_1} B_2 - v[v, B_2] - [B_2, v_1] = 0, \\ L_{31} v - [B_3, v_1] &= v_{x_3} - v_{x_1} B_3 - v[v, B_3] - [B_3, v_1] = 0, \end{aligned} \tag{18}$$

written in terms of the square matrix fields $v(x)$ and $v_1(x)$, where

$$v_1(x) \equiv \int C_{x_1}(\lambda; x) U(\lambda; x)\, \mathrm{d}\Omega(\lambda). \tag{19}$$

Eliminating the extra field v_1 from these two equations, we get the (2 + 1)-dimensional N-wave system (17).

The possibility of deriving integrable systems in these two alternative ways is important, since, while integrable PDEs in 2+1 dimensions (or less) are characterized as the compatibility condition of a linear overdetermined system of PDEs, such a basic property seems to be lost in multidimensions.

(ii) Each linear equation (16) is two dimensional.

2.1.2. Solution space. We now consider the manifold of analytic solutions of equation (17) generated by the dressing procedure. The solutions of equations (10) and (11) can be parametrized as follows:

$$\Phi(\lambda; x) = \int \Phi_0(\lambda, k)\, \mathrm{e}^{kB\cdot x}\, \mathrm{d}k, \tag{20}$$

$$C(\mu; x) = \int \mathrm{e}^{qB\cdot x} C_0(\mu, q)\, \mathrm{d}q, \tag{21}$$

where

$$B \cdot x = \sum_{i=1}^{\dim x} B_i x_i, \tag{22}$$

and where the spectral parameters λ, μ, k, q are scalars. It is simple to see, from the linear limit, that the solution space of equation (17), generated by the dressing algorithm, is full. Indeed, in the linear limit $\Psi(\lambda, \mu) \sim \delta(\lambda - \mu)$ and $U \sim \Phi$. Take $C_0(\lambda, q) = \delta(\lambda - q)$; then the solution v of the three-dimensional N-wave system (17), which in the linear limit reads

$$v(x) \sim \int C(\lambda; x)\Phi(\lambda; x)\, \mathrm{d}\Omega(\lambda) = \int \mathrm{e}^{\lambda B\cdot x} \Phi_0(\lambda, k)\, \mathrm{e}^{kB\cdot x}\, \mathrm{d}k\, \mathrm{d}\Omega(\lambda), \tag{23}$$

is parametrized by the arbitrary matrix function $\Phi_0(\lambda, k)$ of the two scalar spectral parameters λ, k; then its solution space is two dimensional, and therefore it is complete.

2.2. Novelties of the dressing methods introduced in [17]

In [17] we assumed that the kernel of the operator $\hat{\Psi}$ is one dimensional:

$$\dim\ker\hat{\Psi} = 1, \tag{24}$$

i.e., the solution of the homogeneous equation associated with equation (7) is nontrivial:

$$0 = \hat{\Psi} H \quad \Leftrightarrow \quad H(\lambda; x) = U^{(h)}(\lambda; x) f(x), \tag{25}$$

where $U^{(h)}(\lambda; x)$ is some nontrivial solution of the homogeneous equation $\hat{\Psi} H = 0$ and $f(x)$ is an arbitrary matrix function of x. Then the general solution of equation (7) reads

$$U(\lambda; x) = U^{(p)}(\lambda; x) + U^{(h)}(\lambda; x) f(x), \tag{26}$$

where $U^{(p)}(\lambda; x)$ is some particular solution of (7).

As a consequence of the novel assumption (24), equation (12) implies the following equations for U:

$$\mathcal{E}_j(\lambda; x) \equiv L_{j1} U(\lambda; x) - (L_{21} U(\lambda; x)) A^{(j)}(x) = 0, \qquad j = 3, \ldots, \dim x, \tag{27}$$

$$L_{j1} U \equiv U_{x_j} - U_{x_1} B_j - U[v, B_j], \qquad j = 2, \ldots, \dim x, \tag{28}$$

where $A^{(j)}(x)$ are some matrix functions to be defined. We have established that, if $\dim\ker\hat{\Psi} = 1$, then each linear equation (27) for the spectral function $U(\lambda; x)$ is three dimensional.

The associated nonlinear equations are obtained 'saturating the parameter λ' in equations (27) by the integral operator $\int d\Omega(\lambda) C(\lambda; x)\cdot$. In order to express $A^{(j)}(x)$ in terms of U and close the system, we introduce an external dressing function $G(\lambda; x)$, and the associated new matrix fields

$$w^{(00)}(x) \equiv \int G(\lambda; x) U(\lambda; x)\, d\Omega(\lambda), \qquad w^{(j0)}(x) \equiv \int G_{x_j}(\lambda; x) U(\lambda; x)\, d\Omega(\lambda), \quad j > 0, \tag{29}$$

$$w^{(ij)}(x) \equiv \int G_{x_i x_j}(\lambda; x) U(\lambda; x)\, d\Omega(\lambda), \quad i, j > 0, \qquad w^{(ij)}(x) = w^{(ji)}(x). \tag{30}$$

Since the dimensionality of G has no formal restrictions, the above w-fields increase the dimensionality of the nonlinear PDEs. This can be seen, for instance, from their small field limits: $w^{(00)}(x) \sim \int G(\lambda; x)\Phi(\lambda; x)\, d\Omega(\lambda)$, $w^{(j0)}(x) \sim \int G_{x_j}(\lambda; x)\Phi(\lambda; x)\, d\Omega(\lambda)$. Nonlinear equations for these fields appear after applying the integral operators $\int d\Omega(\lambda) G(\lambda; x)\cdot$, $\int d\Omega(\lambda) G_{x_j}(\lambda; x)\cdot$ to equation (27).

To close the system of nonlinear PDEs, one needs (a) equations defining $G(\lambda; x)$ and (b) an additional relation between all the matrix fields, which fixes the arbitrary function $f(x)$ in the solution space (see equation (26)) and may be taken in quite arbitrary form

$$F(v, v^{(1)}, w^{(00)}, w^{(i0)}, w^{(ij)}) = 0, \qquad i, j = 1, 2, \ldots. \tag{31}$$

Let us collect the basic novelties of the algorithm.

(i) The existence of a nontrivial kernel of the basic integral equation implies that the solutions constructed by the dressing depend on an arbitrary function $f(x)$ of the coordinates; this fact has the following important implications.

(ii) The nonlinear system of PDEs constructed by the dressing scheme possesses a distinguished block structure and is underdetermined.

(iii) To close the system and fix its underdeterminacy (or, equivalently, to fix $f(x)$), one has to introduce an 'external and largely arbitrary' relation among the fields (see (31)).

(iv) The system of PDEs depends on two types of matrix fields, those obtained 'saturating the parameter λ' of the solution $U(\lambda; x)$ of the linear integral equation by ingredients of the classical dressing method, whose dimensionality is constrained, and those obtained saturating λ by a novel dressing function $G(\lambda; x)$, whose dimensionality is not constrained. That is why the dimensionality of the solution space, $(n-2)$, can be arbitrarily large.

(v) While integrable PDEs in low dimensions (2+1 or less) are the compatibility of overdetermined systems of linear spectral problems, such a feature seems to be lost for our higher dimensional examples.

A prototype example of the above construction is given by the following four-dimensional system of two matrix equations

$$\mathcal{B}_2(q_1, q_1, q_2)\mathcal{B}_2^{-1}(q_1, q_2, q_3) = \mathcal{B}_3(q_1, q_1, q_2)\mathcal{B}_3^{-1}(q_1, q_2, q_3) = \mathcal{B}_4(q_1, q_1, q_2)\mathcal{B}_4^{-1}(q_1, q_2, q_3) \tag{32}$$

for the three square matrix fields $q_1(x), q_2(x), q_3(x)$, supplemented by the 'largely arbitrary' relation

$$F(q_1, q_2, q_3) = 0 \tag{33}$$

among them, where the matrix blocks $\mathcal{B}_j$ are defined as

$$\mathcal{B}_j(q_1, q_2, q_3) \equiv q_{2x_j} - q_{2x_1}B_j - q_2[q_1, B_j] - [B_j, q_3], \qquad j = 2, 3, 4, \tag{34}$$

and B_j, $j = 2, 3, 4$ are constant diagonal matrices different from the identity. In the simplest case, the largely arbitrary relation (33) can be chosen to be an equation defining one of the fields, say q_3, to be any given function $\gamma(x)$ (in general, a generalized function), interpretable as an 'external arbitrary forcing':

$$F : q_3(x) = \gamma(x). \tag{35}$$

The partially integrable nonlinear PDEs (32)–(35) possess a manifold of analytic solutions of dimension 2.

2.3. Dressing method based on a homogeneous integral equation

Although the starting integral equation used in [17, 18] is still inhomogeneous, the use in [18] of first-order multidimensional dressing operators allows one to enrich the space of analytic solutions of the constructed nonlinear PDEs (although full integrability is not achieved even there), and to simplify the structure of such nonlinear PDEs (which become differential polynomials), if compared to the block structure of equations such as (32)–(35).

In addition, the introduction of such first-order dressing operators makes clear that the inhomogeneous term Φ of the integral equation (7) is not necessary anymore, suggesting the new scenario, discussed in this paper, of a dressing algorithm based on the following *homogeneous* integral equation,

$$0 = \int \Psi(\lambda, \nu; x)U(\nu; x)\, d\Omega(\nu) \equiv \Psi(\lambda, \nu; x) * U(\nu; x), \tag{36}$$

supplemented by the generalized commutation relation

$$\mathcal{A}(\lambda, \nu) * \Psi(\nu, \mu; x) = \Psi(\lambda, \nu; x) * A(\nu, \mu). \tag{37}$$

In the integral equation (36), U is the unknown spectral function depending on the single spectral parameter λ; the kernel Ψ of the integral operator, the dressing function, satisfies the generalized commutation relation (37) for a proper choice of the auxiliary functions $\mathcal{A}$ and A. The function Ψ and, consequently, U, depend on an additional set of variables

$x = (t_1, t_2, \ldots, x_1, x_2, \ldots)$, which are the independent variables of the associated nonlinear PDEs. In this paper we assume that the functions $\mathcal{A}$ and A do not depend on x. In general, all functions are $Q \times Q$ matrices.

Following [17], we assume in this paper that the integral equation (36) possesses nontrivial solutions, namely, that

$$\dim\ker(\Psi *) = d > 0. \tag{38}$$

The general solution of the homogeneous equation (36) reads

$$U(\lambda; x) = \sum_{i=1}^{d} U_h^{(i)}(\lambda; x) f^{(i)}(x), \tag{39}$$

where $U_h^{(i)}$ are independent particular solutions of (36) and $f^{(i)}$ are arbitrary functions of x. It is convenient to introduce a 'rectangular' integral operator in the following way. Let D be a set of points, $D = \{l_1, \ldots, l_M\}$, and let $\mathcal{D}$ be a disjoint set ($\mathcal{D} \cap D = \emptyset$), consisting eventually of continuous and discrete parts. Thus, in the function $\Psi(\lambda, \mu; x)$, $\lambda \in \mathcal{D}$, while $\mu \in \mathcal{D} \cup D$:

$$\Psi(\lambda, \mu; x) = \begin{cases} \psi(\lambda, \mu; x), & \lambda, \mu \in \mathcal{D}, \\ \psi_{0n}(\lambda; x), & \lambda \in \mathcal{D}, \mu = l_n. \end{cases} \tag{40}$$

As a consequence of this assumption, $\lambda \in \mathcal{D} \cup D$ in $U(\lambda; x)$,

$$U(\lambda; x) = \begin{cases} u(\lambda; x), & \lambda \in \mathcal{D}, \\ u_n(x) & \lambda = l_n, \end{cases} \tag{41}$$

and the integral equation (36) reduces to the form

$$\psi(\lambda, \mu; x) * u(\mu; x) + \sum_{i=1}^{M} \psi_{0i}(\lambda; x) u_i(x) = 0, \qquad \lambda \in \mathcal{D}. \tag{42}$$

If the integral operator $\psi(\lambda, \mu; x)*$ is invertible, the solution $u(\lambda; x)$ is uniquely expressed in terms of the M arbitrary functions $u_i(x)$, which may be identified with the functions $f^{(i)}(x)$ in equation (39), and $d = M$. If $\psi(\lambda, \mu; x)*$ is not invertible, then $d > M$. As in [17], we introduce an external dressing function $G(\lambda; x)$ in the next section in order to fix $f^{(i)}(x)$.

We remark that the integral equation (42) can be viewed as an inhomogeneous integral equation with an inhomogeneous term depending on M arbitrary functions. In the rest of the paper we find it more convenient to work with the homogeneous form (36).

The rectangular structure of Ψ implies also different 'square' structures for the integral operators $\mathcal{A}*$ and $*A$ in (37):

$$\mathcal{A}(\lambda, \mu) = \begin{cases} \mathrm{a}(\lambda, \mu), & \lambda, \mu \in \mathcal{D} \\ 0, & \lambda \in D \text{ or } \mu \in D, \end{cases} \tag{43}$$

$$A(\lambda, \mu) = \begin{cases} a(\lambda, \mu), & \lambda, \mu \in \mathcal{D} \\ a_{0m}(\lambda), & \lambda \in \mathcal{D}, \mu = l_m \\ a_{n0}(\mu), & \lambda = l_n, \mu \in \mathcal{D} \\ a_{nm}, & \lambda = l_n, \mu = l_m, \end{cases} \tag{44}$$

where $n, m = 1, \ldots, M$.

To end this section, let us carry out a comparison of the algorithm developed in [17] and in this paper, which we call Alg.1 and Alg.2, respectively.

(1) Both algorithms use the nontrivial kernel of the integral operators, but the integral equation is inhomogeneous in Alg.1 and homogeneous in Alg.2.

(2) Both algorithms use two types of dressing functions: external and internal dressing functions. However, Alg.1 uses two internal dressing functions, $\Phi(\lambda, \mu)$ and $C(\mu; x)$ with the kernel Ψ expressed through Φ and C, while Alg.2 uses the single internal dressing function $\Psi(\lambda, \mu;)$, the kernel of the integral operator.
(3) Both algorithms use two disjoint domains on the spectral parameter plane: a continuous $\mathcal{D}$ and a discrete D. But the number of points M in D is minimized in Alg.2, while this number is an arbitrary $M > \dim\ker\hat{\Psi}$ in Alg.1.
(4) Alg.2 describes the full solution space of the appropriate n-dimensional nonlinear PDEs, while Alg.1 describes only an $(n-2)$ dimensional subspace of analytic solutions.
(5) Both algorithms deal with nonlinear PDEs which may not be considered as a necessary compatibility condition of some overdetermined linear system of PDEs for the associated spectral function.
(6) Both systems of PDEs derived by Alg.1. and Alg.2 have infinitely many commuting flows.

In this paper we concentrate on the case $M = 1$. Before giving more details regarding the solvability of equations (36) and (37), we present the derivation of equation (1).

3. First-order quasilinear PDEs

3.1. Derivation of the PDEs (1)

As usual in the dressing philosophy, the x-dependence of the spectral function $U(\lambda; x)$ is introduced through the x-dependence of the dressing functions. In this paper, as well as in the dressing algorithm introduced in [17], we have two types of dressing functions. The internal dressing function $\Psi(\lambda, \mu; x)$, appearing in the integral equation (36), and an external dressing function $G(\lambda; x)$, $\lambda \in \mathcal{D} \cup D$:

$$G(\lambda; x) = \begin{cases} g(\lambda; x), & \lambda \in \mathcal{D}, \\ g_1(x) & \lambda = l_1, \end{cases} \tag{45}$$

whose role will be explained below.

This x-dependence is defined by the equations

$$\Psi_t(\lambda, \mu; x) + \sum_{j=1}^{N} \mathcal{A}^{(j)}(\lambda, \nu) * \Psi_{x_j}(\nu, \mu; x) = 0, \tag{46}$$

$$G_t(\lambda; x) + \sum_{j=1}^{N} G_{x_j}(\nu; x) * A^{(j)}(\nu, \lambda) = -TG(\nu; x) * A^{(0)}(\nu; \lambda), \tag{47}$$

supplemented by the generalized commutation relations

$$\mathcal{A}^{(j)}(\lambda, \nu) * \Psi(\nu, \mu; x) = \Psi(\lambda, \nu; x) * A^{(j)}(\nu, \mu), \tag{48}$$

where T is an arbitrary constant matrix. Since $\mathcal{A}^{(j)}*$ and $*A^{(j)}$ satisfy the same relation as $\mathcal{A}*$ and $*A$, it follows that they are expressed as functions of the operators $\mathcal{A}*$ and $*A$ respectively,

$$\mathcal{A}^{(j)}* \equiv \rho^{(j)}(\mathcal{A})*, \qquad *A^{(j)} \equiv *\rho^{(j)}(A), \tag{49}$$

where $\rho^{(j)}(\cdot)$ are scalar functions representable as positive power series:

$$\rho^{(j)}(y) = \sum_{k=0}^{\infty} c_k^{(j)} y^k \quad \Rightarrow \quad \begin{cases} *A^{(j)} = *\rho^{(j)}(A) = \sum_{k=0}^{\infty} c_k^{(j)} * \underbrace{A * \cdots * A}_{k} \\ \mathcal{A}^{(j)}* = \rho^{(j)}(\mathcal{A})* = \sum_{k=0}^{\infty} c_k^{(j)} \underbrace{\mathcal{A} * \cdots * \mathcal{A}}_{k} *. \end{cases} \tag{50}$$

From the above definitions and from (44) it follows that

$$A^{(j)}(\lambda,\mu)=\begin{cases}a^{(j)}(\lambda,\mu), & \lambda,\mu\in\mathcal{D},\\ a^{(j)}_{01}(\lambda), & \lambda\in\mathcal{D},\mu=l_1,\\ a^{(j)}_{10}(\mu), & \lambda=l_1,\mu\in\mathcal{D},\\ a^{(j)}_{11}, & \lambda=l_1,\mu=l_1.\end{cases} \tag{51}$$

Applying the operator $\mathcal{A}*$ to equation (36) and using (37), one shows that $A*U$ is another solution of the integral equation (36), i.e.,

$$\Psi*(A*U)=0. \tag{52}$$

Applying instead the operator $\left(\partial_t+\sum_{i=1}^N\mathcal{A}^{(i)}\partial_{x_i}*\right)$ to equation (36) and using the generalized commutation relation (37) and equation (46), one obtains a third solution of the integral equation (36):

$$\Psi(\lambda,\mu;x)*\hat{L}U(\mu;x)=0,\qquad \hat{L}U(\mu;x)\equiv U_t(\mu;x)+\sum_{j=1}^N A^{(j)}(\mu,\nu)*U_{x_j}(\nu;x). \tag{53}$$

In this section we assume that the three solutions U, $A*U$ and $\hat{L}U$ of the integral equation (36) belong to the same one-dimensional matrix subspace spanned by U, i.e.,

$$A(\lambda,\nu)*U(\nu;x)=U(\lambda;x)\tilde{F}(x) \tag{54}$$

$$U_t(\lambda;x)+\sum_{j=1}^N A^{(j)}(\lambda,\nu)*U_{x_j}(\nu;x)=U(\lambda;x)F(x), \tag{55}$$

where the matrices $\tilde{F}$ and F do not depend on the spectral parameters.

As in the dressing scheme introduced in [17], in order to fix $\tilde{F}$ and F, we use the external dressing function G, together with the additional relation

$$G(\lambda;x)*U(\lambda;x)=I, \tag{56}$$

where I is the identity matrix.

Applying $G*$ to equation (54) and using (56), one obtains that

$$\tilde{F}(x)=w(x), \tag{57}$$

where

$$w(x)\equiv G(\lambda;x)*A(\lambda,\mu)*U(\mu;x) \tag{58}$$

so that

$$A(\lambda,\nu)*U(\nu;x)=U(\lambda;x)w(x). \tag{59}$$

In addition, applying repeatedly $A*$ to equation (59), we obtain

$$\rho(A)*U=U\rho(w), \tag{60}$$

where $\rho:\mathbb{R}\to\mathbb{R}$ is any scalar analytic function. Therefore equation (55) becomes

$$U_t(\lambda;x)+\sum_{j=1}^N(U(\lambda;x)\rho^{(j)}(w))_{x_j}=U(\lambda;x)F(x). \tag{61}$$

Applying now $G*$ to (61) and using (56) and (60), one obtains

$$F(x)=\sum_{j=1}^N(\rho^{(j)}(w))_{x_j}+T\rho^{(0)}(w) \tag{62}$$

so that the system (54), (55) takes the final form

$$A(\lambda,\nu)*U(\nu;x)=U(\lambda;x)w(x),$$
$$U_t(\lambda;x)+\sum_{j=1}^{N}U_{x_j}(\lambda;x)\rho^{(j)}(w)=U(\lambda;x)T\rho^{(0)}(w). \tag{63}$$

It is a system of overdetermined linear equations for the spectral function U but, as we shall see in section 6, its role is different from that played by the usual Lax pair for soliton equations. The main feature of the linear equation (63*a*) is that it does not involve x-derivatives.

Applying $G*A*$ to (63*b*), one finally obtains the matrix equation

$$w_t+\sum_{j=1}^{N}w_{x_j}\rho^{(j)}(w)=[w,T\rho^{(0)}(w)] \tag{64}$$

reported in the introduction as equation (1). We recall that w may be either a scalar or a matrix.

3.2. Basic properties of equation (64)

The matrix first-order quasilinear PDE (64), isolated by the above dressing construction, possesses important properties and can be integrated by simple spectral means [21]. Here we summarize, for completeness, some of these properties.

(1) We first observe that, if w solves equation (64), then w^T solves the transposed equation

$$w_t^T+\rho^{(j)}(w^T)\sum_{j=1}^{N}w_{x_j}^T=[\rho^{(0)}(w^T)T^T,w^T], \tag{65}$$

where the order of matrix multiplication is inverted. We also observe that T can be chosen to be diagonal without loss of generality; indeed, if it were not diagonal, the diagonalizing similarity transformation for T would leave equation (64) invariant, just transforming w according to the same similarity transformation.

(2) Let T be a diagonal matrix, $e^{(i)}$ and $\underline{v}^{(i)}$, $i=1,\dots,Q$ be the eigenvalues and the right eigenvectors of the matrix w, suitably normalized by the conditions $v_i^{(i)}=1, i=1,\dots,Q$; introduce the matrix E of the eigenvalues: $E=\mathrm{diag}\,(e^{(1)},\dots,e^{(Q)})$, and the matrix V having the eigenvectors $\underline{v}^{(i)}$ as columns $\left(V_{ij}=v_i^{(j)}\right)$. Then, if w evolves according to equation (64) the spectrum of w evolves in the following simple way:

$$e_t^{(k)}+\sum_{j=1}^{N}e_{x_j}^{(k)}\rho^{(j)}(e^{(k)})=0,\qquad k=1,\dots,Q, \tag{66}$$

$$V_t+\sum_{j=1}^{N}V_{x_j}\rho^{(j)}(E)=[V,T]\rho^{(0)}(E). \tag{67}$$

It follows from (66) that each eigenvalue of w evolves separately according to the scalar version of (64), and is constant along the characteristic straight lines

$$x_j=\rho^{(j)}(e^{(k)})t+\eta_j,\qquad j=1,\dots,n, \tag{68}$$

and η_j are arbitrary integration constants. Therefore it is defined by the implicit equation

$$e^{(k)}=\epsilon^{(k)}(\eta_1,\dots,\eta_N)=\epsilon^{(j)}(x_1-\rho^{(1)}(e^{(k)})t,\dots,x_N-\rho^{(N)}(e^{(k)})t), \tag{69}$$

where $\epsilon^{(k)} : \mathbb{R}^n \to \mathbb{R}, k = 1, \ldots, Q$, are arbitrary functions. Once the eigenvalues are constructed, then V satisfies the linear equation (67), with coefficients depending on the corresponding eigenvalues.

(3) *First reduction*. From equation (66) it follows that the condition

$$e^{(i)}(x) = \text{const}, \qquad i = 1, \ldots, Q, \tag{70}$$

represents a symmetry reduction of the system (64). In this case, equation (67) becomes a linear system of $(Q^2 - Q)$ equations with constant coefficients; therefore the transformation from w to its spectrum linearizes the flow.

(4) *Second reduction*. Consider the subspace of $Q \times Q$ matrices spanned by the basis $\{\omega_0, \ldots, \omega_{Q-1}\}$ given by

$$\omega_0 = I, \quad (\omega_1)_{ij} = \delta_{i+1,j}, \ldots, \quad (\omega_k)_{ij} = \delta_{i+k,j}, \ldots, \quad (\omega_{N-1})_{ij} = \delta_{i+N-1,j}, \quad \text{mod } Q. \tag{71}$$

This subspace is left invariant under matrix multiplication, since

$$\omega_j \omega_k = \omega_k \omega_j = \omega_{j+k}, \qquad \text{mod } Q. \tag{72}$$

Therefore it defines a reduction of (64) from the Q^2 components of w to the Q scalar coefficients $\nu_k, k = 1, \ldots, Q$, of the expansion

$$w = \sum_{k=1}^{Q} \nu_k \omega_{k-1}. \tag{73}$$

In particular, if $Q = 2, N = 1, \rho^{(1)}(y) = y$ and $\rho^{(0)} = 0$, equation (64) reduces to the following particular case of the gas dynamics equations [20]

$$\nu_{1t} + \nu_1 \nu_{1x_1} + \nu_2 \nu_{2x_1} = 0, \qquad \nu_{2t} + \nu_2 \nu_{1x_1} + \nu_1 \nu_{2x_1} = 0. \tag{74}$$

(5) The matrix equation (65) can be written as a vector system of the following form

$$\mathbf{w}_t + \sum_{i=1}^{N} C^{(i)}(\mathbf{w}) \mathbf{w}_{x_i} = \mathbf{B}(\mathbf{w}) \tag{75}$$

for the Q^2-dimensional vector $\mathbf{w}$, where

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_Q \end{bmatrix}, \quad w = (\mathbf{w}_1 \cdots \mathbf{w}_Q), \quad \mathbf{w}_i = \begin{bmatrix} w_{1i} \\ \vdots \\ w_{Qi} \end{bmatrix}, \quad C^{(i)}(\mathbf{w}) = \begin{bmatrix} \rho^{(i)}_{11} I & \cdots & \rho^{(i)}_{Q1} I \\ \vdots & \vdots & \vdots \\ \rho^{(i)}_{1Q} I & \cdots & \rho^{(i)}_{QQ} I \end{bmatrix},$$

$$\mathbf{B}(\mathbf{w}) = \begin{bmatrix} \mathbf{B}_1(\mathbf{w}) \\ \vdots \\ \mathbf{B}_Q(\mathbf{w}) \end{bmatrix}, \quad \mathbf{B}_k(\mathbf{w}) = \begin{bmatrix} [\rho^{(n)}(w^T)T^T, w^T]_{k1} \\ \vdots \\ [\rho^{(n)}(w^T)T^T, w^T]_{kQ} \end{bmatrix},$$

and I is the $Q \times Q$ identity matrix. By construction, the $e^{(j)}$'s are eigenvalues of all matrices $C^{(i)}$ with multiplicity Q. Thus the method of integration of vector first-order quasilinear equations developed in [22–24] may not be applied, at least in the form presented in these references.

Hereafter T is a *diagonal* constant matrix.

4. The general solution of equation (64)

In this section we construct, using the dressing method introduced in this paper, the general solution of the matrix equation (64), which turns out to be characterized by a nonlinear system of non-differential equations for the components of matrix w in the following way.

Proposition 1. *Let $F_{ij} : \mathbb{R}^N \to \mathbb{R}, i, j = 1, \ldots, Q$, be Q^2 arbitrary scalar functions, representable by positive power series, so that $F_{ij}(M_1, \ldots, M_N)$ are well defined matrix functions, where $M_1, \ldots, M_N$ are arbitrary $Q \times Q$ matrices. Let $\{T_1, \ldots, T_Q\}$ be the elements of the constant diagonal matrix T. Then the general solution of the matrix PDE (64) is characterized by the following system of $2Q^2$ non-differential equations:*

$$w_{\alpha\beta} = \sum_{\delta=1}^{Q} \sum_{\gamma_1,\gamma_2=1}^{Q} \left((u_1(x))^{-1}\right)_{\alpha\delta} \left(F_{\delta\gamma_1}(x_1 I - \rho^{(1)}(w)t, \ldots, x_N I - \rho^{(N)}(w)t)\right)_{\gamma_2\beta} (u_1(x))_{\gamma_1\gamma_2}, \tag{76}$$

$$\sum_{\gamma=1}^{Q} (u_1(x))_{\alpha\gamma} (\mathrm{e}^{-\rho^{(0)}(w)T_\alpha t})_{\gamma\beta} = \delta_{\alpha\beta}, \alpha, \beta = 1, \ldots, Q, \tag{77}$$

for the components of the matrix solution $w(x)$ and of the auxiliary matrix function $u_1(x)$.

Remarks.
(1). If $T = 0$, equation (77) gives $u_1 = I$; then equation (76) reduces to the following system of Q^2 non-differential equations for the components of w:

$$w_{\alpha\beta} = \sum_{\gamma=1}^{Q} (F_{\alpha\gamma}(x_1 I - \rho^{(1)}(w)t, \ldots, x_N I - \rho^{(N)}(w)t))_{\gamma\beta}, \qquad \alpha, \beta = 1, \ldots, Q. \tag{78}$$

This equation can also be constructed using equations (66) and (67) for $T = 0$ [21].
(2) If, at last, one is interested in the scalar version of equation (64), its general solution, characterized by the scalar version of (78),

$$w = F(x_1 - \rho^{(1)}(w)t, \ldots, x_N - \rho^{(N)}(w)t), \tag{79}$$

is also easily obtainable using the method of characteristics.
(3) Due to the presence of the Q^2 arbitrary scalar functions $F_{ij}, i, j = 1 \ldots, Q$, the above non-differential equations characterize the general solution of the matrix PDE (64). In particular, if one is interested in solving the Cauchy problem in $\mathbb{R}^N$ with the prescribed initial condition $w_0(x_1, \ldots, x_N) \equiv w(x_1, \ldots, x_N, 0)$, equation (77), evaluated at $t = 0$, implies that $u_1|_{t=0} = I$. Then equation (76) at $t = 0$ implies that

$$F(x_1, \ldots, x_N) = w_0(x_1, \ldots, x_N). \tag{80}$$

The functions F_{ij} being known, the nonlinear non-differential equations (76), (77) allow one to construct, $\forall\, t$, the solution $w(x_1, \ldots, x_N, t)$.

4.1. Proof of proposition 1

To prove proposition 1, we make use of the main ingredients of the dressing scheme: equations (54), (57) and (56), written explicitly using (41), (44) and (45):

$$\begin{aligned} &a(\lambda, \mu) * u(\mu; x) + a_{01}(\lambda)u_1(x) = u(\lambda; x)w(x), && \lambda \in \mathcal{D}, \\ &a_{10}(\mu) * u(\mu; x) + a_{11}u_1(x) = u_1(x)w(x), && \lambda = l_1, \end{aligned} \tag{81}$$

$$g(\mu; x) * u(\mu; x) + g_1(x)u_1(x) = I; \tag{82}$$

equation (58) for w:

$$w(x) \equiv G(\lambda; x) * A(\lambda, \mu) * U(\mu; x) = g(\lambda; x) * a(\lambda, \mu) * u(\mu; x) \\ + g(\lambda; x) * a_{01}(\lambda)u_1(x) + g_1(x)(a_{10}(\lambda) * u(\lambda; x) + a_{11}u_1(x)); \tag{83}$$

and the generalized commutation relation (37), which we rewrite to emphasize the fact that the parameters λ and μ take values in different domains:

$$\mathcal{A}(\lambda, \nu) * \Psi(\nu, \mu) = \Psi(\lambda, \nu) * A(\nu, \mu), \qquad \lambda \in \mathcal{D}, \qquad \mu \in \mathcal{D} \cup D. \tag{84}$$

Using formulae (40), (41), (43) and (44), equation (84) takes the form

$$\mathrm{a}(\lambda, \nu) * \psi(\nu, \mu; x) = \psi(\lambda, \nu; x) * a(\nu, \mu) + \psi_{01}(\lambda; x)a_{10}(\mu), \qquad \lambda, \mu \in \mathcal{D}, \tag{85}$$

$$\mathrm{a}(\lambda, \nu) * \psi_{01}(\nu; x) = \psi(\lambda, \nu; x) * a_{01}(\nu) + \psi_{01}(\lambda; x)a_{11}, \qquad \lambda \in \mathcal{D}, \mu = l_1. \tag{86}$$

This system should be viewed as a system of linear equations for the functions ψ and ψ_{01}, where a, a, a_{ij} are largely arbitrary. The following two choices for a and a have been explored so far:

$$(1) \quad \mathrm{a}(\lambda, \mu) = a(\lambda)\delta(\lambda - \mu), \qquad \mathrm{a}(\lambda, \mu) = a(\lambda)\delta(\lambda - \mu), \tag{87}$$

$$(2) \quad \mathrm{a}(\lambda, \mu) = \mathrm{i}\delta'(\lambda - \mu), \qquad \mathrm{a}(\lambda, \mu) = -\mathrm{i}\delta'(\lambda - \mu), \tag{88}$$

where δ and δ' are the Dirac function and its derivative.

It is possible to verify that the first choice (87) corresponds to the case in which the dressing function Ψ does not depend on x and, consequently, the eigenvalues of w are constant as well. Therefore the choice (87) leads to the trivial reduction discussed in the previous section (see (70)).

As we shall see in the following, the second choice (88) allows one to capture instead the general solution of the matrix PDE (64).

4.1.1. Solution space associated with equations (88). Construction of the dressing function $\Psi(\lambda, \mu; x)$. Under assumption (88), the function Ψ is completely defined by equation (46), which reads

$$\psi_t(\lambda, \mu; x) + \sum_{j=1}^{N} \rho^{(j)}(\mathrm{i}\partial_\lambda)\psi_{x_j}(\lambda, \mu; x) = 0, \qquad \lambda, \mu \in \mathcal{D}, \tag{89}$$

$$\psi_{01_t}(\lambda; x) + \sum_{j=1}^{N} \rho^{(j)}(\mathrm{i}\partial_\lambda)\psi_{01_{x_j}}(\lambda; x) = 0, \qquad \lambda \in \mathcal{D}, \mu = l_1, \tag{90}$$

and by the system (85), (86),

$$\mathrm{i}\psi_\lambda(\lambda, \mu; x) = \mathrm{i}\psi_\mu(\lambda, \mu; x) + \psi_{01}(\lambda; x)a_{10}(\mu), \qquad \lambda, \mu \in \mathcal{D}, \tag{91}$$

$$\mathrm{i}\psi_{01_\lambda}(\lambda; x) = \psi(\lambda, \nu; x) * a_{01}(\nu) + \psi_{01}(\lambda; x)a_{11}, \qquad \lambda \in \mathcal{D}, \mu = l_1. \tag{92}$$

Equations (89)–(92) suggest to represent ψ, ψ_{01} and a_{10} in the Fourier forms

$$\psi(\lambda, \mu; x) = \int_{-\infty}^{\infty} \mathrm{d}\varkappa \int_{-\infty}^{\infty} \mathrm{d}q \int_{\mathbb{R}^N} \mathrm{d}k\, \tilde{\psi}(\varkappa, q, k) \exp\left(\mathrm{i}\varkappa\lambda + iq\mu + \mathrm{i}\sum_{j=1}^{N} k_j[x_j - \rho^{(j)}(-\varkappa)t]\right), \tag{93}$$

$$\psi_{01}(\lambda; x) = \int_{-\infty}^{\infty} \mathrm{d}\varkappa \int_{\mathbb{R}^N} \mathrm{d}k\, \tilde{\psi}_{01}(\varkappa, k) \exp\left(\mathrm{i}\varkappa\lambda + \mathrm{i}\sum_{j=1}^{N} k_j[x_j - \rho^{(j)}(-\varkappa)t]\right), \tag{94}$$

$$a_{10}(\mu) = \int_{-\infty}^{\infty} \tilde{a}_{10}(q)\,\mathrm{e}^{\mathrm{i}q\mu}\,\mathrm{d}q, \tag{95}$$

$$u(\lambda; x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{2\pi}\tilde{u}(q; x)\,\mathrm{e}^{\mathrm{i}q\lambda}, \tag{96}$$

where $k = (k_1, \ldots, k_N)$.

For future convenience, we also represent $a_{01}(\mu)$ via the contour integral

$$a_{01}(\mu) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{01}} \tilde{a}_{01}(q)\,\mathrm{e}^{\mathrm{i}q\mu}\,\mathrm{d}q, \tag{97}$$

where Γ_{01} is a sufficiently large contour containing all singularities of the integrand.

With the above representations, equations (89), (90) are automatically satisfied, while equations (91), (92) yield the relations

$$\tilde{\psi}(\varkappa, q, k)(\varkappa - q) = -\tilde{\psi}_{01}(\varkappa, k)\tilde{a}_{10}(q), \tag{98}$$

$$\tilde{\psi}_{01}(\varkappa, k)(\varkappa + a_{11}) = -\int_{-\infty}^{\infty} \mathrm{d}\mu \int_{-\infty}^{\infty} \mathrm{d}q\, \tilde{\psi}(\varkappa, q, k)\,\mathrm{e}^{\mathrm{i}q\mu} a_{01}(\mu). \tag{99}$$

We solve the first of these equations with respect to $\tilde{\psi}$

$$\tilde{\psi}(\varkappa, q, k) = -\frac{\tilde{\psi}_{01}(\varkappa, k)\tilde{a}_{10}(q)}{\varkappa - q} + \phi(\varkappa, k)\delta(\varkappa - q), \tag{100}$$

where ϕ is an arbitrary function of its arguments, and substitute the result in (99):

$$\begin{aligned} &\tilde{\psi}_{01}(\varkappa, k)(\varkappa + a_{11} - \eta(\varkappa)) = -\int_{-\infty}^{\infty} \mathrm{d}\mu\phi(\varkappa, k)\,\mathrm{e}^{\mathrm{i}\varkappa\mu} a_{01}(\mu), \\ &\eta(\varkappa) = \int_{-\infty}^{\infty} \mathrm{d}\mu \int_{-\infty}^{\infty} \mathrm{d}q \frac{\tilde{a}_{10}(q)\,\mathrm{e}^{\mathrm{i}q\mu} a_{01}(\mu)}{\varkappa - q}. \end{aligned} \tag{101}$$

We take the following solution of this equation:

$$\begin{aligned} &\tilde{\psi}_{01}(\varkappa, k) = -\phi(\varkappa, k) \int_{-\infty}^{\infty} \mathrm{d}\mu\, \mathrm{e}^{\mathrm{i}\varkappa\mu} a_{01}(\mu)\Omega^{-1}(\varkappa), \\ &\Omega(\varkappa) \equiv \varkappa + a_{11} - \eta(\varkappa); \end{aligned} \tag{102}$$

then equation (100) yields

$$\tilde{\psi}(\varkappa, q, k) = \phi(\varkappa, k)\left[\int_{-\infty}^{\infty} \mathrm{d}\mu \frac{\mathrm{e}^{\mathrm{i}\varkappa\mu} a_{01}(\mu)\Omega^{-1}(\varkappa)\tilde{a}_{10}(q)}{\varkappa - q} + \delta(\varkappa - q)\right]. \tag{103}$$

Substituting $\tilde{\psi}$ and $\tilde{\psi}_{01}$ into (42), with $M = 1$, and applying the operator $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}\xi\lambda}\cdot$ to the result, we obtain

$$\begin{aligned} \tilde{\phi}(-\xi, x_1 - \rho^{(1)}(\xi)t, \ldots, x_N - \rho^{(N)}(\xi)t)&\left[\tilde{u}(\xi; x) + \left(\int_{-\infty}^{\infty} \mathrm{d}\nu\, a_{01}(\nu)\,\mathrm{e}^{-\mathrm{i}\xi\nu}\right)\Omega^{-1}(-\xi)\right. \\ &\left.\times\left(\int_{-\infty}^{\infty} \frac{\mathrm{d}q}{q - \xi}\tilde{a}_{10}(-q)\tilde{u}(q; x) - u_1(x)\right)\right] = 0, \end{aligned} \tag{104}$$

where

$$\tilde{\phi}(-\xi, x_1 - \rho^{(1)}(\xi)t, \dots, x_N - \rho^{(N)}(\xi)t) = \int_{\mathbb{R}^N} \phi(-\xi, k) \exp\left(\mathrm{i} \sum_{j=1}^{N} k_j (x_j - \rho^{(j)}(\xi)t) \right) \mathrm{d}k. \tag{105}$$

We also remark that equation (81) takes the form

$$-\mathrm{i}u_\lambda(\lambda; x) + a_{01}(\lambda)u_1(x) = u(\lambda; x)w(x), \qquad \lambda \in \mathcal{D}, \tag{106}$$

$$a_{10}(\mu) * u(\mu; x) + a_{11}u_1(x) = u_1(x)w(x), \qquad \lambda = l_1. \tag{107}$$

Due to equation (97), the solution of (106) leads to the following representation of u,

$$u(\lambda; x) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{01}} \mathrm{d}q \tilde{a}_{01}(q) \,\mathrm{e}^{\mathrm{i}q\lambda} u_1(x)(w(x) - qI)^{-1}, \tag{108}$$

different from the Fourier representation (96). Then equation (107) yields

$$\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\mu \oint_{\Gamma_{01}} \mathrm{d}q a_{10}(\mu)\tilde{a}_{01}(q) \,\mathrm{e}^{\mathrm{i}q\mu} u_1(x)(w(x) - q)^{-1} = u_1(x)w(x) - a_{11}u_1(x). \tag{109}$$

The formulae derived so far are applicable to both the scalar and the matrix equations. Below we consider these two cases separately, starting with the simpler one.

Scalar nonlinear PDEs. In the scalar version

$$w_t + \sum_{j=1}^{N} w_{x_j} \rho^{(j)}(w) = 0 \tag{110}$$

of (64) the RHS is zero. In this case we can set $T = 0$; therefore equation (47) admits the trivial solution $g = 0$, $g_1 = 1$ so that $u_1 = 1$ (see (82)).

It follows that equation (83) coincides with equation (107), while equation (109) reads

$$\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\mu \oint_{\Gamma_{01}} \mathrm{d}q \frac{a_{10}(\mu)\tilde{a}_{01}(q) \,\mathrm{e}^{\mathrm{i}q\mu}}{w - q} = w - a_{11}. \tag{111}$$

This equation indicates that, if a_{10} and a_{01} are given, then (111) is an algebraic constraint for w, which would imply that w is constant. To avoid such a trivialization, equation (111) must be taken as a definition of a_{10} or a_{01}, and w must be considered as the independent variable.

In addition, one should make sure that condition (111) does not coincide with the condition $\Omega(\varkappa) = 0$ (see (102*b*)), i.e.,

$$\eta(\varkappa) \equiv \int_{-\infty}^{\infty} \mathrm{d}\mu \int_{-\infty}^{\infty} \mathrm{d}q \frac{\tilde{a}_{10}(q) \,\mathrm{e}^{\mathrm{i}q\mu} a_{01}(\mu)}{\varkappa - q} \neq \varkappa + a_{11}; \tag{112}$$

otherwise equations (102)–(104) would make no sense.

A possible (and simple) choice for a_{01} is given by

$$\tilde{a}_{01}(q) = \frac{1}{q - b}, \tag{113}$$

implying

$$a_{01}(\mu) = \mathrm{e}^{\mathrm{i}b\mu}. \tag{114}$$

Then, assuming that $\tilde{a}_{10}(q)$ is an entire function, equation (111) gives

$$\tilde{a}_{10}(w) = -\frac{(w + a_{11})(w + b)}{2\pi} + \tilde{a}_{10}(-b), \tag{115}$$

while equation (112) reads

$$\eta(\varkappa) = 2\pi \frac{\tilde{a}_{10}(-b)}{\varkappa + b} \neq \varkappa + a_{11} \Rightarrow \Omega(\varkappa) \neq 0. \tag{116}$$

Hereafter we assume, without loss of generality, that

$$\tilde{a}_{10}(-b) = 0 \quad \Rightarrow \quad \eta(\varkappa) = 0, \qquad \Omega(\varkappa) = \varkappa + a_{11}. \tag{117}$$

Now we rewrite equation (104) using equations (111)–(115):

$$\tilde{\phi}(-\xi, x_1 - \rho^{(1)}(\xi)t, \ldots, x_N - \rho^{(N)}(\xi)t) \times \left[\tilde{u}(\xi; x) - \left(\int_{-\infty}^{\infty} \mathrm{d}q \frac{(q-b)(q-a_{11})}{(a_{11}-\xi)(q-\xi)} \tilde{u}(q; x) + \frac{2\pi}{a_{11}-\xi}\right) \delta(\xi - b)\right] = 0. \tag{118}$$

Using again equations (114), equation (108) yields u in terms of w,

$$u(\lambda; x) = \frac{\mathrm{e}^{\mathrm{i}b\lambda} - \mathrm{e}^{\mathrm{i}w\lambda}}{w - b}, \tag{119}$$

so that

$$\tilde{u}(q; x) = 2\pi \frac{\delta(q-b) - \delta(q-w)}{w-b}. \tag{120}$$

Substituting it into equation (118) one gets

$$\frac{\delta(w-\xi)}{w-b} \tilde{\phi}(-\xi, x_1 - \rho^{(1)}(\xi)t, \ldots, x_N - \rho^{(N)}(\xi)t) = 0, \tag{121}$$

which is satisfied iff

$$\tilde{\phi}(-w, x_1 - \rho^{(1)}(w)t, \ldots, x_N - \rho^{(N)}(w)t) = 0. \tag{122}$$

The implicit equation (122) for w suggests to take function $\tilde{\phi}(y, x_1, \ldots, x_N)$ in the form

$$\tilde{\phi}(y, x_1, \ldots, x_N) = y + F(x_1, \ldots, x_N). \tag{123}$$

Thus equation (122) yields

$$w = F(x_1 - \rho^{(1)}(w)t, \ldots, x_N - \rho^{(N)}(w)t), \tag{124}$$

which is the well-known non-differential equation defining implicitly the solution of the Cauchy problem in $\mathbb{R}^N$

$$w_t + \sum_{j=1}^{N} w_{x_j} \rho^{(j)}(w) = 0, \qquad w|_{t=0} = F(x_1, \ldots, x_N) \tag{125}$$

for the scalar version of equation (64).

Then the direct problem, the mapping from $w|_{t=0}$ to the dressing function ψ (or $\tilde{\phi}$, via (102) and (103)), is simply given by

$$w|_{t=0} = F(x_1, \ldots, x_N) \quad \Rightarrow \quad \tilde{\phi}(y, x_1, \ldots, x_N) = y + F(x_1, \ldots, x_N). \tag{126}$$

From the inverse problem point of view, given $\tilde{\phi}$ from the initial data through equations (122), (123), the spectral function $\tilde{u}$ is reconstructed solving equation (118), which is equivalent to

$$\tilde{u}(\xi; x) - \int_{-\infty}^{\infty} \frac{\mathrm{d}q\, \mathrm{d}\nu}{2\pi} \mathrm{e}^{\mathrm{i}\nu(b-\xi)} \frac{(q-b)(q-a_{11})}{(a_{11}-\xi)(q-\xi)} \tilde{u}(q; x) = \frac{2\pi}{a_{11}-\xi} \delta(\xi - b) + \alpha(x)\delta(w - \xi), \tag{127}$$

where $\alpha(x)$ is found requiring that $\tilde{u}$ be compatible with expression (120). This request fixes the value $\alpha = -2\pi/(w-b)$.

Matrix nonlinear PDEs. In the matrix case, we choose the operators $\mathcal{A}*$ and $*A$ to be scalar, i.e., $a_{10}(\lambda)$ and $a_{01}(\lambda)$ are scalar functions and a_{11} is a scalar parameter. Now we consider the general case $u_1 \neq I$. Then equation (108) yields

$$u(\lambda; x) = \frac{u_1(x)}{2\pi \mathrm{i}} \oint_{\Gamma_{01}} \mathrm{d}q\, \tilde{a}_{01}(q) \Big(w(x) - qI\Big)^{-1} \mathrm{e}^{\mathrm{i}q\lambda}. \tag{128}$$

Assuming the invertibility of u_1, equation (109) yields the following matrix generalization of (111):

$$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\mu \oint_{\Gamma_{01}} \mathrm{d}q\, a_{10}(\mu)\tilde{a}_{01}(q)\, \mathrm{e}^{\mathrm{i}q\mu}(w(x) - qI)^{-1} = w(x) - a_{11}I. \tag{129}$$

The expressions for η, $\tilde{a}_{01}(q)$ and $a_{01}(\mu)$ remain the same (see equations (112)–(117)).

We also assume that w is diagonalizable with eigenvalues $\{e^{(j)}, \ldots, e^{(Q)}\}$, and that $V(x)$ is the matrix of right eigenvectors, so that

$$w = V \operatorname{diag}(e^{(1)}, \ldots, e^{(Q)}) V^{-1}. \tag{130}$$

Then, applying V^{-1} and V respectively from the left and from the right to equation (129), we obtain the scalar equation (111), in which w is replaced by its eigenvalues. Consequently, also the expression of $\tilde{a}_{10}$ is given by (115), and (117) holds too.

The matrix generalizations of equations (118), (119) and (120) read, respectively,

$$\begin{aligned}&\tilde{\phi}(-\xi, x_1 - \rho^{(1)}(\xi)t, \ldots, x_N - \rho^{(N)}(\xi)t) \\ &\times \left[\tilde{u}(\xi; x) - \left(\int_{-\infty}^{\infty} \mathrm{d}q \frac{(q-b)(q-a_{11})}{(a_{11}-\xi)(q-\xi)} \tilde{u}(q; x) + \frac{2\pi}{a_{11}-\xi} u_1(x)\right) \delta(\xi - b)\right] = 0,\end{aligned} \tag{131}$$

$$u(\mu; x) = u_1(x)V(x)(\mathrm{e}^{\mathrm{i}b\mu I} - \mathrm{e}^{\mathrm{i}\mu E})(E - bI)^{-1}V^{-1}(x) = u_1(x)(\mathrm{e}^{\mathrm{i}b\mu I} - \mathrm{e}^{\mathrm{i}\mu w})(w - bI)^{-1}, \tag{132}$$

and

$$\begin{aligned}\tilde{u}(q; x) &= 2\pi u_1(x)V(x)(I\delta(q-b) - \delta(qI - E))(E - bI)^{-1}V^{-1}(x) \\ &= 2\pi u_1(x)(I\delta(q-b) - \delta(qI - w))(w - bI)^{-1}.\end{aligned} \tag{133}$$

Substituting (133) into the integral equation (131) and multiplying the result by the matrix $(w - bI)$ from the right, one obtains the matrix distribution equation:

$$\tilde{\phi}(-\xi, x_1 - \rho^{(1)}(\xi)t, \ldots, x_N - \rho^{(N)}(\xi)t)u_1(x)\delta(\xi I - w) = 0. \tag{134}$$

Since $\delta(\xi I - w) = V\delta(\xi I - E)V^{-1}$, equation (134) is equivalent to the distribution equation

$$\begin{aligned}&\sum_{\gamma_1=1}^{Q}\sum_{\gamma_2=1}^{Q}\sum_{\gamma_3=1}^{Q} \tilde{\phi}_{\alpha\gamma_1}(-\xi, x_1 - \rho^{(1)}(\xi)t, \ldots, x_N - \rho^{(N)}(\xi)t) \\ &\qquad \times (u_1(x))_{\gamma_1\gamma_2} V_{\gamma_2\gamma_3}(x)\delta(\xi - \mathrm{e}^{(\gamma_3)})V^{-1}_{\gamma_3\beta}(x) = 0,\end{aligned} \tag{135}$$

which implies

$$\begin{aligned}&\sum_{\gamma_1=1}^{Q}\sum_{\gamma_2=1}^{Q}\sum_{\gamma_3=1}^{Q} \tilde{\phi}_{\alpha\gamma_1}(-\mathrm{e}^{(\gamma_3)}, x_1 - \rho^{(1)}(\mathrm{e}^{(\gamma_3)})t, \ldots, x_N - \rho^{(N)}(\mathrm{e}^{(\gamma_3)})t) \\ &\qquad \times (u_1(x))_{\gamma_1\gamma_2} V_{\gamma_2\gamma_3}(x)V^{-1}_{\gamma_3\beta}(x) = 0, \qquad \alpha, \beta = 1, \ldots, Q.\end{aligned} \tag{136}$$

Choosing, in analogy with the scalar case, the following form for the matrix $\tilde{\phi}$:

$$\tilde{\phi}_{ij}(y, x_1, \ldots, x_N) = y\delta_{ij} + F_{ij}(x_1, \ldots, x_N), \tag{137}$$

and eliminating eigenvalues and eigenvectors, equation (136) becomes

$$(u_1 w)_{\alpha\beta} = \sum_{\gamma_1=1}^{Q}\sum_{\gamma_2=1}^{Q}(u_1(x))_{\gamma_1\gamma_2}\big(F_{\alpha\gamma_1}(x_1 I - \rho^{(1)}(w)t, \ldots, x_N I - \rho^{(N)}(w)t)\big)_{\gamma_2\beta}. \tag{138}$$

Applying u_1^{-1} from the left we finally obtain the non-differential equation (76), involving the solution $w(x)$ of (64) and the auxiliary matrix function $u_1(x)$.

4.1.2. The dressing function $G(\lambda; x)$ and the construction of $u_1(x)$. If $T \neq 0$, the solution of (64) is computed involving also functions u_1 and G, which are related by equation (82). Substituting u given by (132) into (82) and using the following Fourier representation for $g(\mu; x)$:

$$g(\mu; x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\varkappa}{2\pi}\, \hat{g}(\varkappa; x)\, \mathrm{e}^{\mathrm{i}\varkappa\mu}, \tag{139}$$

one obtains

$$\begin{aligned}\sum_{\gamma_1=1}^{Q}\sum_{\gamma_2=1}^{Q}&[(\hat{g}_{\alpha\gamma_1}(-b; x)I - \hat{g}_{\alpha\gamma_1}(-w; x))(w - bI)^{-1}]_{\gamma_2\beta}(u_1(x))_{\gamma_1\gamma_2}\\ &+ (g_1(x)u_1(x))_{\alpha\beta} = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, \ldots, Q.\end{aligned} \tag{140}$$

This is a determined system of linear equations for the elements of u_1, where functions g and g_1 are solutions of equation (47), which reads, in terms of (44), (45) and (51),

$$\begin{aligned}g_t(\lambda; x) &+ \sum_{j=1}^{N} g_{x_j}(\nu; x) * a^{(j)}(\nu, \lambda) + \sum_{j=1}^{N} g_{1x_j}(x) a_{10}^{(j)}(\lambda)\\ &= -Tg(\nu; x) * a^{(0)}(\nu, \lambda) - Tg_1(x)a_{10}^{(0)}(\lambda), \qquad \lambda \in \mathcal{D},\end{aligned} \tag{141}$$

$$\begin{aligned}g_{1t}(x) &+ \sum_{j=1}^{N} g_{1x_j}(x) a_{11}^{(j)} + \sum_{j=1}^{N} g_{x_j}(\mu; x) * a_{01}^{(j)}(\mu)\\ &= -Tg_1(x)a_{11}^{(0)} - Tg(\mu; x) * a_{01}^{(0)}(\mu), \qquad \lambda = l_1.\end{aligned} \tag{142}$$

The representations of functions $A^{(j)}$, $j = 0, 1, 2, \ldots,$ in terms of A can be obtained recursively through the formulae

$$A^{(n)}(\lambda, \nu) * A(\nu, \mu) = \begin{cases} a^{(n)}(\lambda, \nu) * a(\nu, \mu) + a_{01}^{(n)}(\lambda)a_{10}(\mu), & \lambda, \mu \in \mathcal{D},\\ a_{01}^{(n)}(\lambda)a_{11} + a^{(n)}(\lambda, \nu) * a_{01}(\nu), & \lambda \in \mathcal{D}, \mu = l_1,\\ a_{10}^{(n)}(\nu) * a(\nu, \mu) + a_{11}^{(n)} a_{10}(\mu), & \lambda = l_1, \mu \in \mathcal{D},\\ a_{10}^{(n)}(\nu) * a_{01}(\nu) + a_{11}^{(n)} a_{11}, & \lambda = \mu = l_1.\end{cases} \tag{143}$$

Due to the choice (88) for a, and to the representations (95), (113), (114) of a_{10} and a_{01}, it is possible to look for a solution of the system (141), (142) satisfying the following properties,

$$\begin{aligned}&g_1 = 0,\; g(\lambda; x) * a_{01}^{(j)}(\lambda) = 0,\\ &g(\mu; x) * a^{(j)}(\mu, \lambda) = g(\mu; x) * \rho^{(j)}(a(\mu, \lambda)) \equiv \rho^{(j)}(\mathrm{i}\partial_\lambda)g(\lambda; x), \qquad j = 1, 2, \ldots,\end{aligned} \tag{144}$$

which reduce such a system to the single equation

$$g_t(\lambda; x) + \sum_{j=1}^{N} \rho^{(j)}(\mathrm{i}\partial_\lambda) g_{x_j}(\lambda; x) + T\rho^{(0)}(\mathrm{i}\partial_\lambda)g(\lambda; x) = 0. \tag{145}$$

Indeed, as it is possible to verify using equation $a_{01}(\mu) = \exp(\mathrm{i}b\mu)$ and formulae (143), the restrictions (144) are simply satisfied by imposing the single condition $\hat{g}(-b; x) = 0$.

Then the matrix function g satisfying (144), (145) can be represented in the following Fourier form:

$$\begin{aligned} g(\lambda; x) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\varkappa}{2\pi} \int_{\mathbb{R}^N} \mathrm{d}k \int_{\Gamma_\omega} \mathrm{d}\omega\, \tilde{g}(\varkappa, k, \omega) \exp\left(\mathrm{i}\varkappa\lambda + \mathrm{i}\sum_{j=1}^{N} k_j x_j - \mathrm{i}\omega t\right), \\ \tilde{g}_{\alpha\beta}(\varkappa, k, \omega) &= (\varkappa + b) h_{\alpha\beta}(k)\delta\left(\omega - \sum_{j=1}^{N} k_j \rho^{(j)}(-\varkappa) + \mathrm{i}\rho^{(0)}(-\varkappa)T_\alpha\right) \end{aligned} \tag{146}$$

where Γ_ω is any contour passing through the support of the Dirac function, and $h_{\alpha\beta}, \alpha, \beta = 1, \ldots, Q$ are arbitrary scalar functions of their arguments.

Then replacing

$$\hat{g}(\varkappa; x) = \int_{\mathbb{R}^N} \mathrm{d}k \int_{\Gamma_\omega} \mathrm{d}\omega\, \tilde{g}(\varkappa, k, \omega) \exp\left(\mathrm{i}\sum_{j=1}^{N} k_j x_j - \mathrm{i}\omega t\right) \tag{147}$$

in equation (140), one obtains

$$\sum_{\gamma_1,\gamma_2=1}^{Q} [H_{\alpha\gamma_1}(x_1 I - \rho^{(1)}(w)t, \ldots, x_N I - \rho^{(N)}(w)t)\, \mathrm{e}^{-\rho^{(0)}(w)T_\alpha t}]_{\gamma_2\beta}(u_1(x))_{\gamma_1\gamma_2} = \delta_{\alpha\beta}, \tag{148}$$

where

$$H_{\alpha\beta}(x_1, \ldots, x_N) \equiv \int_{\mathbb{R}^N} \mathrm{d}k\, h_{\alpha\beta}(k) \exp\left(\mathrm{i}\sum_{l=1}^{N} k_l x_l\right). \tag{149}$$

If we choose $h_{\alpha\beta}(\xi, k) = \left(\prod_j \delta(k_j)\right)\delta_{\alpha\beta}, \Rightarrow H_{\alpha\beta} = \delta_{\alpha\beta}$, then equation (148) reduces to (77). This simplification can be done without loss of generality since, if it were not made, it would correspond to the following redefinition of the arbitrary matrix function F appearing in (76):

$$F \to HFH^{-1}. \tag{150}$$

5. Second-order quasilinear PDEs

5.1. Derivation of second-order PDEs

To increase the order of the nonlinear PDEs, we introduce the x-dependence through the next equations

$$\Psi_{t_m}(\lambda, \mu; x) + \sum_{j=1}^{N} \mathcal{A}^{(mj)}(\lambda, \nu) * \Psi_{x_j}(\nu, \mu; x) = 0, \tag{151}$$

$$G_{t_m}(\lambda, q; x) + \sum_{j=1}^{N} G_{x_j}(\nu, q; x) * A^{(mj)}(\nu, \lambda) = -q^m S^{(m)} G(\nu, q; x) * A^{(m0)}(\nu, \lambda), \tag{152}$$

supplemented by the generalized commutation relation

$$\mathcal{A}^{(mj)}(\lambda, \nu) * \Psi(\nu, \mu; x) = \Psi(\lambda, \nu; x) * A^{(mj)}(\nu, \mu), \tag{153}$$

where q is a new spectral parameter, $S^{(m)}$ are constant diagonal matrices,

$$\mathcal{A}^{(mj)}* = \rho^{(mj)}(\mathcal{A})*, \qquad *A^{(mj)} = *\rho^{(mj)}(A), \tag{154}$$

and $\rho^{(mj)} : \mathbb{R} \to \mathbb{R}$ are arbitrary analytic functions.

Together with the field w introduced in the previous section, we introduce also the fields

$$v^{(n)}(x) = (G(\mu, q; x)q^n) * U(\mu; x), \qquad n = 1, 2, \ldots, \qquad v(x) \equiv v^{(1)}(x), \tag{155}$$

where now '$*$' means integration also over q.

Applying the operators $\left(\partial_{t_m} + \sum_{j=1}^N \mathcal{A}^{(mj)}\partial_{x_j} *\right)$ to (36), one gets

$$\Psi(\lambda, \mu) * \left(U_{t_m}(\mu; x) + \sum_{j=1}^{N} A^{(mj)}(\mu, \nu) * U_{x_j}(\nu, \mu; x) \right) = 0. \tag{156}$$

Assuming, as before, that the solutions of (36) belong to the one-dimensional matrix subspace generated by U, the spectral equations are similar to (54), (55):

$$A(\lambda, \nu) * U(\nu; x) = U(\lambda; x)\tilde{F}(x), \tag{157}$$

$$U_{t_m}(\lambda; x) + \sum_{j=1}^{N} A^{(mj)}(\lambda, \nu) * U_{x_j}(\nu, \mu; x) = U(\lambda; x)F^{(m)}(x). \tag{158}$$

Applying $G*$ to (157) and (158) and using condition (56) and equations (152), (60), we obtain the expression of $\tilde{F}(x)$ and $F^{(m)}(x)$ in terms of the matrix fields w and $v^{(n)}$:

$$\tilde{F}(x) = w(x), \qquad F^{(m)}(x) = \sum_{j=1}^{N} (\rho^{(mj)}(w))_{x_j} + S^{(m)}v^{(m)}(x)\rho^{(m0)}(w). \tag{159}$$

Therefore equations (157) and (158) become the following overdetermined system for the spectral function $U(\lambda; x)$:

$$\begin{aligned} &A(\lambda, \mu)U(\mu; x) = U(\lambda; x)w(x), \\ &U_{t_m}(\lambda; x) + \sum_{j=1}^{N} U_{x_j}(\lambda; x)\rho^{(mj)}(w) = U(\lambda; x)S^{(m)}v^{(m)}(x)\rho^{(m0)}(w). \end{aligned} \tag{160}$$

In order to construct the nonlinear PDE for w, we apply $G * A*$ to equations (160b) and use equations (60), (151) and (155), obtaining

$$w_{t_m} + \sum_{j=1}^{N} w_{x_j}\rho^{(mj)}(w) = [w, S^{(m)}v^{(m)}\rho^{(m0)}(w)]. \tag{161}$$

To write the equations for $v^{(k)}$, we apply instead $(Gq^k)*$ to equation (160b), obtaining

$$v^{(k)}_{t_m} + \sum_{j=1}^{N} v^{(k)}_{x_j}\rho^{(mj)}(w) + S^{(m)}v^{(k+m)}\rho^{(m0)}(w) = v^{(k)}S^{(m)}v^{(m)}\rho^{(m0)}(w). \tag{162}$$

To construct a complete system of nonlinear PDEs, consider the three equations:

$$\begin{aligned} &v^{(1)}_{t_1} + \sum_{j=1}^{N} v^{(1)}_{x_j}\rho^{(1j)}(w) + S^{(1)}v^{(2)}\rho^{(10)}(w) = v^{(1)}S^{(1)}v^{(1)}\rho^{(10)}(w), \\ &v^{(2)}_{t_1} + \sum_{j=1}^{N} v^{(2)}_{x_j}\rho^{(1j)}(w) + S^{(1)}v^{(3)}\rho^{(10)}(w) = v^{(2)}S^{(1)}v^{(1)}\rho^{(10)}(w), \\ &v^{(1)}_{t_2} + \sum_{j=1}^{N} v^{(1)}_{x_j}\rho^{(2j)}(w) + S^{(2)}v^{(3)}\rho^{(20)}(w) = v^{(1)}S^{(2)}v^{(2)}\rho^{(20)}(w), \end{aligned} \tag{163}$$

corresponding to $(m, k) = (1, 1), (1, 2), (2, 1)$.

We may choose, without loss of generality, $S^{(1)} = 1$, $S^{(2)} = S$. Then, expressing $v^{(2)}$ and $v^{(3)}$ in terms of $v = v^{(1)}$,

$$\begin{aligned} v^{(2)} &= v^2 - \mathcal{L}_1(v)(\rho^{(10)}(w))^{-1}, \\ v^{(3)} &= v^{(2)}v - \mathcal{L}_1(v^{(2)})(\rho^{(10)}(w))^{-1}, \end{aligned} \tag{164}$$

one obtains a single matrix equation of the second-order for v and w,

$$\begin{aligned} \mathcal{L}_2(v)\rho^{(10)}(w) + S\mathcal{L}_1(\mathcal{L}_1(v)(\rho^{(10)}(w))^{-1})\rho^{(20)}(w) &= S\mathcal{L}_1(v^2)\rho^{(20)}(w) \\ &+ [S\mathcal{L}_1(v)\rho^{(20)}(w), v] + S\mathcal{L}_1(v)[(\rho^{(10)}(w))^{-1}v, \rho^{(20)}(w)\rho^{(10)}(w)] \\ &+ [v, Sv^2]\rho^{(20)}(w)\rho^{(10)}(w), \end{aligned} \tag{165}$$

where the differential operators $\mathcal{L}_m$, $m = 1, 2$ are defined by

$$\mathcal{L}_m(f(x)) = f_{t_m}(x) + \sum_{i=1}^{N} f_{x_i}(x)\rho^{(mi)}(w). \tag{166}$$

A complete system of second-order matrix PDEs in arbitrary dimensions for the fields w and v is obtained coupling equation (165) with equation (161) for $m = 2$, obtaining

$$\begin{aligned} &\mathcal{L}_2(w) = [w, S(v^2 - \mathcal{L}_1(v))v(\rho^{(10)}(w))^{-1}], \\ &\mathcal{L}_2(v)\rho^{(10)}(w) + S\mathcal{L}_1(\mathcal{L}_1(v)(\rho^{(10)}(w))^{-1})\rho^{(20)}(w) = S\mathcal{L}_1(v^2)\rho^{(20)}(w) \\ &\qquad + [S\mathcal{L}_1(v)\rho^{(20)}(w), v] + S\mathcal{L}_1(v)[(\rho^{(10)}(w))^{-1}v, \rho^{(20)}(w)\rho^{(10)}(w)] \\ &\qquad + [v, Sv^2]\rho^{(20)}(w)\rho^{(10)}(w). \end{aligned} \tag{167}$$

But the independent equation (161) for $m = 1$,

$$\mathcal{L}_1(w) = [w, v\rho^{(10)}(w)], \tag{168}$$

is also satisfied by the fields w and v and must be viewed as an integrable constraint for the evolutionary (with respect to the time t_2) system (167).

We remark that the evolutionary system in arbitrary dimensions (167) is completely integrable only under the nonlinear constraint (168).

Here we consider two explicit reductions.

(1) Let

$$N = 2, \qquad \rho^{(11)} = \rho^{(22)} = w, \qquad \rho^{(10)} = \rho^{(20)} = I, \qquad \rho^{(12)} = \rho^{(21)} = 0, \tag{169}$$

then

$$\mathcal{L}_1(f) = f_{t_1} + f_{x_1}w, \qquad \mathcal{L}_2(f) = f_{t_2} + f_{x_2}w, \tag{170}$$

and one obtains the nonlinear system

$$\begin{aligned} &w_{t_2} + w_{x_2}w = \left[w, S\left(v^2 - v_{t_1} - v_{x_1}w\right)\right], \qquad v_{t_2} + v_{x_2}w + S\left(v_{t_1} + v_{x_1}w\right)_{t_1} + S\left(v_{t_1} + v_{x_1}w\right)_{x_1}w \\ &\qquad = S\left((v^2)_{t_1} + (v^2)_{x_1}w\right) + [S(v_{t_1} + v_{x_1}w), v] + [v, Sv^2], \end{aligned} \tag{171}$$

subjected to the constraint

$$w_{t_1} + w_{x_1}w = [w, v]. \tag{172}$$

Using this constraint, equation (171*b*) can be rewritten in the more convenient form

$$\begin{aligned} &v_{t_2} + Sv_{t_1t_1} + v_{x_2}w + Sv_{x_1x_1}w^2 + 2Sv_{x_1t_1}w - [S, v](v_{t_1} + v_{x_1}w) \\ &\qquad - 2S(v_{t_1}v + v_{x_1}vw) + [Sv^2, v] = 0. \end{aligned} \tag{173}$$

(2) Let

$$N = 2, \qquad \rho^{(11)} = \rho^{(22)} = \rho^{(10)} = \rho^{(20)} = I, \qquad \rho^{(12)} = \rho^{(21)} = 0, \tag{174}$$

then

$$\mathcal{L}_1(f) = f_{t_1} + f_{x_1} = f_{\tau_1}, \qquad \mathcal{L}_2(f) = f_{t_2} + f_{x_2} = f_{\tau_2}, \tag{175}$$

and the nonlinear system reads

$$\begin{aligned} &w_{\tau_2} = [w, S(v^2 - v_{\tau_1})], \\ &v_{\tau_2} + Sv_{\tau_1\tau_1} = S(v^2)_{\tau_1} + \left[Sv_{\tau_1}, v\right] + [v, Sv^2], \end{aligned} \tag{176}$$

with the constraint

$$w_{\tau_1} = [w, v]. \tag{177}$$

We remark that, in this reduction, equation (176*b*), which involves only the field v, is the matrix Bürgers equation [26].

Using equations (41), (45) and (155), the fields $v^{(j)}$ have the following spectral representation:

$$v^{(j)}(x) \equiv (G(\mu, q; x)q^j) * U(\mu; x) = \left(g(\mu, q; x)q^j u(\mu; x)\right) + \int g_1(q, x)q^j \, \mathrm{d}q u_1. \tag{178}$$

The dressing functions ψ and G are responsible for the dimensionality of the space of solutions for w and v, respectively.

As we did in section (3.2), we may separate the diagonal and off-diagonal parts of equation (161), obtaining

$$e^{(k)}_{t_m} + \sum_{j=1}^{N} e^{(k)}_{x_j} \rho^{(mj)}(e^{(k)}) = 0, \qquad k = 1, \ldots, Q, \tag{179}$$

$$V_{t_m} + \sum_{j=1}^{N} V_{x_j} \rho^{(mj)}(E) + S^{(m)} v^{(m)} V = VD, \qquad D \text{ diagonal}. \tag{180}$$

Equation (179) coincides with equation (66), i.e., the eigenvalues of w evolve separately according to the scalar version of equation (161). Instead, equation (180) cannot be written in the same form of equation (67), since $v^{(m)}$ is not diagonal matrix. This is a principal difference between equations (161) and (64).

5.2. The solution of the system (167), (168)

In this section we construct, using the dressing method introduced in this paper, the solution of the matrix equations (168) and (167), which turns out to be characterized by a nonlinear system of non-differential equations for the components of the unknown matrices w and v in the following way.

Proposition 2. *Let $F_{ij} : \mathbb{R}^N \to \mathbb{R}, i, j = 1, \ldots, Q$ and $H_{ij} : \mathbb{R}^{N+1} \to \mathbb{R}, i, j = 1, \ldots, Q$ be $2Q^2$, arbitrary scalar functions, representable by positive power series, so that $F_{ij}(M_1, \ldots, M_N)$ and $H_{ij}(q, M_1, \ldots, M_N)$ are well defined matrix functions, where $M_1, \ldots, M_N$ are arbitrary $Q \times Q$ matrices. Then the solutions of the matrix system (167), subjected to the matrix constraint (168), are characterized by the following system of $3Q^2$ non-differential equations:*

$$\begin{aligned} w_{\alpha\beta} = \sum_{\gamma_1,\gamma_2,\delta=1}^{Q} \left(u_1^{-1}(x)\right)_{\alpha\delta} \Bigg(& F_{\delta\gamma_1}\Bigg(x_1 I - \sum_{m=1}^{2} \rho^{(m1)}(w)t_m, \ldots, x_N I \\ & - \sum_{m=1}^{2} \rho^{(mN)}(w)t_m\Bigg)\Bigg)_{\gamma_2\beta} (u_1(x))_{\gamma_1\gamma_2} = 0, \qquad \alpha, \beta = 1, \ldots, Q, \end{aligned} \tag{181}$$

$$\int_{-\infty}^{\infty} \mathrm{d}q \sum_{\gamma_1=1}^{Q}\sum_{\gamma_2=1}^{Q} \Bigg[H_{\alpha\gamma_1}\Bigg(q, x_1 I - \sum_{m=1}^{2}\rho^{(m1)}(w)t_m, \ldots, x_N I - \sum_{m=1}^{2}\rho^{(mN)}(w)t_m\Bigg) \exp\Bigg(-\sum_{m=1}^{2} S_\alpha^{(m)} q^m \rho^{(m0)}(w)t_m\Bigg)\Bigg]_{\gamma_2\beta} (u_1(x))_{\gamma_1\gamma_2}$$
$$= \delta_{\alpha\beta}, \alpha, \beta = 1, \ldots, Q, \tag{182}$$

$$v_{\alpha\beta}(x) = \int_{-\infty}^{\infty} \mathrm{d}q\, q \sum_{\gamma_1,\gamma_2=1}^{Q} \Bigg(H_{\alpha\gamma_1}\Bigg(q, x_1 I - \sum_{m=1}^{2}\rho^{(m1)}(w)t_m, \ldots, x_N I - \sum_{m=1}^{2}\rho^{(mN)}(w)t_m\Bigg) \exp\Bigg(-\sum_{m=1}^{2} S_\alpha^{(m)} q^m \rho^{(m0)}(w)t_m\Bigg)\Bigg)_{\gamma_2\beta} (u_1(x))_{\gamma_1\gamma_2},$$
$$\alpha, \beta = 1, \ldots, Q, \tag{183}$$

for the components of the unknown matrices $w(x)$ *and* $v(x)$*, and of the auxiliary matrix function* $u_1(x)$*.*

Proof of proposition 2. To prove this proposition, we proceed as in sections 3 and 4. The solutions w and $v^{(n)}$ are constructed using the algorithm presented in section 4, but the functions Ψ and G will be now defined by equations (151), (152). Equation (151) yields

$$\begin{aligned} &\psi_{t_m}(\lambda, \mu; x) + \sum_{j=1}^{N}\rho^{(mj)}(\mathrm{i}\partial_\lambda)\psi_{x_j}(\lambda, \mu; x) = 0, \qquad \lambda \in \mathcal{D}, \\ &\psi_{01_{t_m}}(\lambda; x) + \sum_{j=1}^{N}\rho^{(mj)}(\mathrm{i}\partial_\lambda)\psi_{01_{x_j}}(\lambda; x) = 0, \qquad \lambda = l_1, \end{aligned} \tag{184}$$

where $m = 1, 2$. The analysis of the system (184) coincides with the analysis carried out in section 4.1.1 but ψ and ψ_{01} exhibit the following Fourier representations:

$$\psi(\lambda, \mu; x) = \int_{-\infty}^{\infty} \mathrm{d}\varkappa \int_{-\infty}^{\infty} \mathrm{d}q \int_{\mathbb{R}^N} \mathrm{d}k\, \tilde{\psi}(\varkappa, q, k) \times \exp\Bigg(\mathrm{i}\varkappa\lambda + iq\mu + \mathrm{i}\sum_{j=1}^{N} k_j\Bigg(x_j - \sum_{m=1}^{2}\rho^{(mj)}(-\varkappa)t_m\Bigg)\Bigg), \tag{185}$$

$$\psi_{01}\lambda; x) = \int_{-\infty}^{\infty} \mathrm{d}\varkappa \int_{\mathbb{R}^N} \mathrm{d}k\, \tilde{\psi}_{01}(\varkappa, k) \exp\Bigg(\mathrm{i}\varkappa\lambda + \mathrm{i}\sum_{j=1}^{N} k_j\Bigg(x_j - \sum_{m=1}^{2}\rho^{(mj)}(-\varkappa)t_m\Bigg)\Bigg), \tag{186}$$

and equation (76) is replaced by (181). □

The x-dependence of G is introduced by equation (152), with $a(\lambda, \mu)$ given in (88):

$$g_{t_m}(\lambda, q; x) + \sum_{j=1}^{N} g_{x_j}(\nu, q; x) * a^{(mj)}(\nu, \lambda) + \sum_{j=1}^{N} g_{1_{x_j}}(q, x) a_{10}^{(mj)}(\lambda)$$
$$= -S^{(m)}q^m\big(g(\nu, q; x) * a^{(m0)}(\nu, \lambda) + g_1(q, x)a_{10}^{(m0)}(\lambda)\big), \qquad \lambda \in \mathcal{D}$$

$$g_{1t_m}(q,x)+\sum_{j=1}^{N} g_{1x_j}(q,x)a_{11}^{(mj)}+\sum_{j=1}^{N} g_{x_j}(\mu,q;x)*a_{01}^{(mj)}(\mu)$$
$$=-S^{(m)}q^m\big(g_1(q,x)a_{11}^{(m0)}+g(\mu,q;x)*a_{01}^{(m0)}(\mu)\big),\qquad \lambda=l_1, \tag{187}$$

where $m=1,2$. We impose again the conditions

$$g_1(q;x)=0,\qquad g(\mu,q;x)*a_{01}^{(mj)}(\mu)=0,$$
$$g(\mu,q;x)*a^{(mj)}(\mu,\lambda)=g(\mu,q;x)*\rho^{(mj)}(a(\mu,\lambda))=\rho^{(mj)}(\mathrm{i}\partial_\lambda)g(\lambda,q;x),\ j=1,2,\ldots \tag{188}$$

on the solution of the system (187), reducing it to the single equation

$$g_{t_m}(\lambda,q;x)+\sum_{j=1}^{N}\rho^{(mj)}(\mathrm{i}\partial_\lambda)g_{x_j}(\lambda,q;x)+S^{(m)}q^m\rho^{(m0)}(\mathrm{i}\partial_\lambda)g(\lambda,q;x)=0. \tag{189}$$

Functions g satisfying (188), (189) can be written in the following Fourier form:

$$g(\lambda,q;x)=\int_{\mathbb{R}^N}\mathrm{d}k\int_{-\infty}^{\infty}\mathrm{d}\omega\int_{-\infty}^{\infty}\mathrm{d}\varkappa\,\tilde g(\varkappa,q,k,\omega)\exp\left(\mathrm{i}\varkappa\lambda+\mathrm{i}\sum_{l=1}^{N}k_lx_l-\mathrm{i}\sum_{m=1}^{2}\omega^{(m)}t_m\right) \tag{190}$$

where

$$\tilde g_{\alpha\beta}(\varkappa,q,k,\omega)=(\varkappa+b)h_{\alpha\beta}(q,k)\prod_{m=1}^{2}\delta\left(\omega^{(m)}-\sum_{l=1}^{N}k_l\rho^{(ml)}(-\varkappa)+\mathrm{i}S_\alpha^{(m)}q^m\rho^{(m0)}(-\varkappa)\right). \tag{191}$$

Now $\hat g(\varkappa,q;x)$ takes the form

$$\hat g(\varkappa,q;x)=\int_{\mathbb{R}^N}\mathrm{d}k\int_{-\infty}^{\infty}\mathrm{d}\omega\,\tilde g(\varkappa,q,k,\omega)\exp\left(\mathrm{i}\sum_{l=1}^{N}k_lx_l-\mathrm{i}\sum_{m=1}^{2}\omega^{(m)}t_m\right), \tag{192}$$

and the analogue of equation (77) is equation (182), where

$$H(q,x_1,\ldots,x_N)=\int_{\mathbb{R}^N}\mathrm{d}k\,h(q,k)\exp\left(\mathrm{i}\sum_{j=1}^{N}k_jx_j\right). \tag{193}$$

At last, substituting in equation (178) expressions (132) and (190) of u and g, we obtain (183).

Initial-boundary value problem. A well-posed initial-boundary value problem for equations (167), (168) is the Cauchy problem for the system (167), in which the initial conditions for w and v at $t_2=0$ are any pair of matrix functions satisfying the multidimensional PDE (168). This initial constraint can be satisfied assigning arbitrarily v at $t_2=0$ and w at $t_1=t_2=0$:

$$w^{(00)}=w(x)|_{t_1=t_2=0}\qquad v^{(0)}=v(x)|_{t_2=0}. \tag{194}$$

Then, integrating equation (168) with respect to t_1, one obtains $w^{(0)}=w(x)|_{t_2=0}, \forall t_1$. At last, given $v^{(0)}$, $w^{(0)}$, the evolutionary system (167) allows one to construct w and v, $\forall\, t_2$.

The algorithm allowing one to integrate such an initial-boundary value problem consists of three steps.

(1) At $t_1=t_2=0$, the system (181)–(183) reads

$$w^{(00)} = (u^{(00)})^{-1} F(x_1, \dots, x_N) u^{(00)}, \tag{195}$$

$$\tilde{H}(0, x_1, \dots, x_N) u^{(00)} = I, \tag{196}$$

$$v^{(00)} = \tilde{H}'(0, x_1, \dots, x_N) u^{(00)}, \tag{197}$$

where $v^{(00)} = v^{(0)}(x)|_{t_1=0}$ is given, $u^{(00)} = u_1(x)|_{t_1=t_2=0}$, and

$$\begin{aligned} &\tilde{H}(t_1, x_1, \dots, x_N) = \int_{-\infty}^{\infty} \mathrm{d}q\, H(q, x_1, \dots, x_N)\, \mathrm{e}^{-qt_1}, \\ &\tilde{H}'(0, x_1, \dots, x_N) = \tilde{H}_{t_1}(t_1, x_1, \dots, x_N)\big|_{t_1=0}. \end{aligned} \tag{198}$$

This system of three matrix equations must be solved for $F(x_1, \dots, x_N)$, $u^{(00)}$ and $\tilde{H}'(0, x_1, \dots, x_N)$. The function $\tilde{H}(0, x_1, \dots, x_N)$ remains arbitrary.

(2) The system (181)–(183), evaluated at $t_2 = 0$, reads

$$\begin{aligned} w^{(0)}_{\alpha\beta} = \sum_{\gamma_1,\gamma_2,\delta=1}^{Q} &((u^{(0)})^{-1})_{\alpha\delta} \big[F_{\delta\gamma_1}(x_1 I - \rho^{(11)}(w^{(0)})t_1, \dots, x_N I - \rho^{(1N)}(w^{(0)})t_1)\big]_{\gamma_2\beta} (u^{(0)})_{\gamma_1\gamma_2}, \\ &\alpha, \beta = 1, \dots, Q, \end{aligned} \tag{199}$$

$$\begin{aligned} \sum_{\gamma_1=1}^{Q} \sum_{\gamma_2=1}^{Q} &\big[\tilde{H}_{\alpha\gamma_1}(\rho^{(10)}(w^{(0)})t_1, x_1 I - \rho^{(11)}(w^{(0)})t_1, \dots, x_N I \\ &- \rho^{(1N)}(w^{(0)})t_1)\big]_{\gamma_2\beta} (u^{(0)})_{\gamma_1\gamma_2} = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, \dots, Q, \end{aligned} \tag{200}$$

$$\begin{aligned} v^{(0)}_{\alpha\beta} = \sum_{\gamma_1=1}^{Q} \sum_{\gamma_2=1}^{Q} &\big[\tilde{H}'_{\alpha\gamma_1}(\rho^{(10)}(w^{(0)})t_1, x_1 I - \rho^{(11)}(w^{(0)})t_1, \dots, x_N I \\ &- \rho^{(1N)}(w^{(0)})t_1)\big]_{\gamma_2\beta} (u^{(0)})_{\gamma_1\gamma_2}, \qquad \alpha, \beta = 1, \dots, Q, \end{aligned} \tag{201}$$

where $u^{(0)} = u_1(x)|_{t_2=0}$. Equations (200), (201) must be solved for $u^{(0)}$ and $\tilde{H}_{\alpha\gamma_1}(t_1, z_1, \dots, z_N)$. Then equation (199) gives $w^{(0)}$.

(3) After that, the functions w, u_1 and v can be constructed as the solutions of the non-differential system (181)–(183), $\forall\, t_1, t_2$.

6. Auxiliary linear system

We have seen in the previous sections that the dressing algorithm produces, together with the integrable nonlinear PDEs in arbitrary dimensions, also the associated linear overdetermined system of equations for the spectral function U, whose coefficients are related to the fields of the nonlinear PDEs.

In this section we show that, *unlike the classical S-integrable case, the integrability condition for the overdetermined system for the spectral function U does not fix completely the integrable nonlinear PDEs*. This is not surprising, since the derivation of the nonlinear PDEs of this paper requires, together with the linear system for U, also the external dressing function G and the constraint (56) for it.

Consider the following linear system, which corresponds to the general form of the nonlinear PDEs treated in this paper:

$$E^{(0)} := A(\lambda, \mu) * U(\mu; x) = U(\lambda; x) w(x), \tag{202}$$

$$E^{(m)} := U_{t_m}(\lambda; x) + \sum_{j=1}^{N} U_{x_j}(\lambda; x)\rho^{(mj)}(x) = U(\lambda; x)F^{(m)}(x) \tag{203}$$

for the arbitrary matrix coefficients w, $\rho^{(im)}$ and $F^{(m)}$. The compatibility of equation (203) with (202), for any fixed m, yields

$$\sum_{j=1}^{N} U_{x_j}[w, \rho^{(mj)}] + U\left[w_{t_m} + \sum_{j=1}^{N} w_{x_j}\rho^{(mj)} - [w, F^{(m)}]\right] = 0. \tag{204}$$

Assuming that U and its derivatives are independent matrix functions, equation (204) implies the following relations among the coefficients of the system (203), (202):

$$[w, \rho^{(mj)}] = 0, \tag{205}$$

$$w_{t_m} + \sum_{j=1}^{N} w_{x_j}\rho^{(mj)} - [w, F^{(m)}] = 0. \tag{206}$$

In the scalar case, equation (205) is identically satisfied, so that equation (206) is some relation among $\rho^{(mj)}, w$ and $F^{(m)}$ for any m. Thus, the system (205), (206) cannot be considered as a complete system of nonlinear equations for some fields.

In the matrix case, equation (205) implies that the $\rho^{(mj)}$'s are arbitrary functions of w, representable in a power series of w with scalar coefficients depending arbitrarily on x. The particular case in which these coefficients are independent of x is in agreement with the form of the nonlinear PDEs derived in this paper, but no condition is imposed on the coefficients $F^{(m)}$. Therefore, also in this case, equations (205), (206) cannot be considered as a complete system of nonlinear equations.

The compatibility of two equations from the list (203), for instance, $E^{(m)}$ and $E^{(n)}$, $n \neq m$, instead yields

$$\begin{aligned} &\sum_{i=1}^{N}\sum_{j=1}^{N} U_{x_i x_j}[\rho^{(nj)}, \rho^{(mi)}] + \sum_{j=1}^{N} U_{x_j}\left[\sum_{i=1}^{N}\left(\rho^{(nj)}_{x_i}\rho^{(mi)} - \rho^{(mj)}_{x_i}\rho^{(ni)}\right)\right. \\ &\qquad \left. + \left(\rho^{(nj)}_{t_m} - \rho^{(mj)}_{t_n}\right) - [F^{(n)}, \rho^{(mj)}] + [F^{(m)}, \rho^{(nj)}]\right] \\ &\qquad + U\left[F^{(m)}_{t_n} - F^{(n)}_{t_m} + \sum_{i=1}^{N}\left(F^{(m)}_{x_i}\rho^{(ni)} - F^{(n)}_{x_i}\rho^{(mi)}\right) + [F^{(n)}, F^{(m)}]\right] = 0. \end{aligned} \tag{207}$$

Assuming again the independence of U and its derivatives, one obtains the following relations among the coefficients:

$$\begin{aligned} &[\rho^{(nj)}, \rho^{(mi)}] = 0, \\ &\rho^{(nj)}_{t_m} + \sum_{i=1}^{N} \rho^{(nj)}_{x_i}\rho^{(mi)} - [\rho^{(nj)}, F^{(m)}] = n \leftrightarrow m, \\ &F^{(n)}_{t_m} + \sum_{i=1}^{N} F^{(n)}_{x_i}\rho^{(mi)} - F^{(n)}F^{(m)} = n \leftrightarrow m. \end{aligned} \tag{208}$$

Equation (208a) follows from the relations (205), and is satisfied if, for instance, $\rho^{(ni)}$ are functions of w, as has been previously established. Then equation (208b) is a consequence of the relation (206). Equation (208c) prescribes relations among the $F^{(m)}$'s, but leaving one of them free. Therefore the system (205), (206), (208) cannot be considered as a complete system of nonlinear PDEs for the coefficients of the linear system (202), (203).

7. Conclusions

Using a new version of the dressing method, based on a homogeneous integral equation with a nontrivial kernel, we constructed a new type of integrable multidimensional nonlinear PDEs. There are no formal restrictions on the dimensionality of the PDEs, while these restrictions are very severe in the case of the classical S-integrable systems, which are known to be the first examples of nonlinear PDEs treatable by dressing technics. Several modifications and extensions of the dressing algorithm presented here will be considered in subsequent papers.

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